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The necessary conditions for the existence of an additional integral in the problem of the motion of a heavy ellipsoid on a smooth horizontal plane☆

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ABSTRACT

The equations of motion of an ellipsoid on a smooth horizontal plane are similar to the equations of motion of a heavy rigid body with a fixed point. In general, one integral is also lacking in order to integrate them. For a triaxial ellipsoid, the centre of mass of which coincides with the geometrical centre, it is proved that an additional integral is lacking (in the generic case).

1. Formulation of the problem

The equations of motion and their first integrals. Consider the motion of a heavy rigid body, having the shape of a triaxial ellipsoid, on a smooth horizontal plane. One holonomic constraint is imposed on the system, namely, the height of the centre of mass above the plane is uniquely defined by the orientation of the body, i.e., the system has five degrees of freedom. Suppose OXYZ is a fixed system of coordinates, S is the centre of mass of the body, Se_1, Se_2 and Se_3 are the principal central axes of inertia of the body, $\mathbf{v} = (\upsilon_x, \upsilon_y, \upsilon_z)$ is the velocity vector of the centre of mass of the body, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector of the body, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector of the ascending vertical, directed along OZ, m is the mass of the body, (b_1, b_2, b_3) are the principal semiaxes of the ellipsoid, $\mathbf{J} = \mathrm{daig}(J_1, J_2, J_3)$ is the principal central inertia tensor of the body, $\mathbf{r}(\gamma)$ is the radius vector of the centre of mass of the body at the point where it touches the plane, $\mathbf{N}$ is the normal reaction of the plane and $z = -\langle \mathbf{r}(\gamma), \gamma \rangle$ is the elevation of the centre of mass of the body above the plane.

The law of motion of the centre of mass has the form

$$m\dot{\mathbf{v}} = \mathbf{N} - mg\boldsymbol{\gamma}$$

In the projection onto the OX and OY axes we obtain $\dot{\upsilon}_x = \dot{\upsilon}_y = 0$, i.e., we can always choose an inertial system of coordinates with centre at the centre of the body, moving uniformly along a horizontal plane and in which $v_x = v_y = 0$. In the projection onto the OZ axis we have

$$\upsilon_z = \dot{z}, \quad m\ddot{z} = N - mg$$

Hence, expressing the value of N and substituting the result into the law of variation of the principal angular momentum about S and taking into account the condition for the vertical unit vector to be constant, we obtain the Euler–Poisson equations

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{r} \times (m\ddot{z} + mg)\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0 \tag{1.1}$$

(here and henceforth we assume that $z = z(\gamma)$).

Equations (1.1) for any parameters have three first integrals (the energy integral, the area integral and the geometrical integral):

$$H = \frac{\langle \mathbf{J}\boldsymbol{\omega}, \boldsymbol{\omega} \rangle}{2} + \frac{m\dot{z}^2}{2} + mgz, \quad K = \langle \mathbf{J}\boldsymbol{\omega}, \boldsymbol{\gamma} \rangle, \quad \Gamma = \langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle$$

The limitation of system (1.1) to the common level of integrals

$$M_{k,c} = \{(\mathbf{M}, \boldsymbol{\gamma}) \mid K = k, \Gamma = c\}$$

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is a Lagrange system with two degrees of freedom. For it to be integrable, according to Liouville's theorem, a single additional integral is lacking.

The purpose of this paper is to determine the necessary conditions for an additional integral to exist.

2. A triaxial ellipsoid

Suppose the principal central axes of inertia are codirectional with the principal axes of the body surface, while the centre of mass coincides with the geometrical centre. Then

$$\mathbf{r}(\boldsymbol{\gamma}) = -\langle \mathbf{B}\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle^{-1/2}\mathbf{B}\boldsymbol{\gamma}, \quad \mathbf{B} = \mathrm{diag}(b_1^2, b_2^2, b_3^2), \quad z = \sqrt{b_1^2\gamma_1^2 + b_2^2\gamma_2^2 + b_3^2\gamma_3^2}$$

Suppose $\mathbf{M} = (M_1, M_2, M_3)$ is the angular momentum; then the equations of motion (1.1) can be written in the form of the Hamilton system

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}} \tag{2.1}$$

where

$$H = \frac{1}{2}\langle \mathbf{AM}, \mathbf{M}\rangle + mgz,$$

$$\mathbf{A}(\boldsymbol{\gamma}) = D^{-1}\begin{Vmatrix} \xi_{23} & \eta_{123} & \eta_{132} \\ \eta_{123} & \xi_{31} & \eta_{231} \\ \eta_{132} & \eta_{231} & \xi_{12} \end{Vmatrix}$$

$$\xi_{ij} = J_iJ_j + m(J_ia_j^2 + J_ja_i^2), \quad \eta_{ijk} = -ma_ia_jJ_k; \quad i, j, k = 1, 2, 3$$

$$D = J_1J_2J_3 + m(J_1J_2a_3^2 + J_1J_3a_2^2 + J_2J_3a_1^2)$$

$$a_1 = (b_3^2 - b_2^2)\gamma_2\gamma_3 z^{-1} \quad (123)$$

We will consider Eqs (2.1) in the complex domain. We will seek an additional integral F in the class of meromorphic functions.

To prove the complex integrability the following considerations are necessary.

Suppose we have a system of differential equations with right-hand side holomorphic and analytical in ε (ε is a small parameter)

$$\dot{\mathbf{x}} = \mathbf{f}^{(0)}(\mathbf{x}) + \varepsilon\mathbf{f}^{(1)}(\mathbf{x}) + \varepsilon^2\mathbf{f}^{(2)}(\mathbf{x}) + \ldots, \quad \mathbf{x} \in \mathbb{C}^n \tag{2.2}$$

We will assume that it admits of $k \le n-1$ integrals, represented in the form

$$\mathbf{F} = \mathbf{F}^{(0)} + \varepsilon\mathbf{F}^{(1)} + \varepsilon^2\mathbf{F}^{(2)} + \ldots, \quad \mathbf{F} \in \mathbb{C}^k \tag{2.3}$$

Suppose **I** are plane integrals of the unperturbed system, corresponding to the zero value of ε in expansion (2.2).

The following theorem, which, in the idealogical plan, goes back to the work of Kovalevskaya and Lyapunov, is well known.

Theorem 1 (Refs 1, 2, Ref. 2, p. 245, Appendix (4.11)).

1°. Suppose the function $f^{(0)}(x)$ is quasi-homogeneous, i.e., the unperturbed system is invariant under the tranformations $t \to \delta t$, $x_j \to \delta^{p_j} x_j$, and it has a non-zero solution

$$x_j^{(0)}(t) = \alpha_j t^{p_j}, \quad \alpha_j \in \mathbb{C}, \quad p_j \in \mathbb{Z}, \quad j = 1, \ldots, n$$

2°. The integrals $F^{(0)}(x)$ are functionally independent quasi-homogeneous polynomials in this solution (i.e., the rank of the Jacobi matrix for the functions $F^{(0)}(x)$ is a maximum in the solution).

3°. The functions $\mathbf{f}^{(i)}(\mathbf{x})$,(i = 1,2,. . .) are polynomails.

Then, the following assertions hold:

1) the Kovalevskaya matrix

$$\tilde{K} = \frac{\partial f^{(0)}}{\partial x}(\alpha) - \mathrm{diag}(\mathbf{p}) \text{ is disgonalizable;} \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \mathbf{p} = (p_1, p_2, \ldots, p_n)$$

and its eigenvalues ρ_i, called Kovalevskaya exponents, are integer;

Suppose β_i are eigenvectors of the Kovalevskaya matrix.

2) the solution of system (2.2) has the form

$$\mathbf{x}(t) = \mathbf{x}^{(0)}(t) + \varepsilon\mathbf{x}^{(1)}(t) + \varepsilon^2\mathbf{x}^{(2)}(t) + \ldots$$

where

$$\mathbf{x}^{(i)}(t) = \begin{cases} \sum\limits_{j=0}^{i-\ell+1} s_{ij}(t)\ln^j(t), & \text{if} \quad i \geq \ell - 1 \\ s_{i0}(t), & \text{if} \quad i < \ell - 1 \end{cases} \tag{2.4}$$

and $s_{ij}(t)$ are the sums of monomials in t, and ℓ is a parameter, which corresponds to the term of least order in ε, where a logarithmic correction is required (if relation (2.4) does not contain logarithms, in general we assume that $\ell = \infty$).

3) the following k conditions are satisfied

$$\left\langle \frac{\partial \mathbf{I}}{\partial \mathbf{x}}(\mathbf{x}^{(0)}), \quad \mathbf{s}_{\ell 1} \right\rangle = 0 \tag{2.5}$$

Remarks. Assertion 1) is Yoshida's theorem[1] which follows from conditions 1° and 2°. Assertion 2) is derived by the method of variation of arbitrary constants, taking into account the scale-invariance of the equations and conditions 3°. Assertion 3) is obtained by substituting expansion (2.4) into integrals (2.3) and expanding them in series in powers of ε. Where algorithms are first encountered there will be branching of the solution, which contradicts the uniqueness of the first integrals. It follows form condition 2° that the Jacobian of the replacement F = F(I) exists and is non-degenerate ($\det(\partial\mathbf{F}/\partial\mathbf{I}) \neq 0$) in the solution, and hence we can consider relations for I instead of F. Hence, the required assertion follows.

The results of applying Theorem 1 to the problem considered can be formulated in the form of the following theorem.

Theorem 2 (this problem is formulated in Ref. 3). Suppose all the J_1, J_2 and J_3 are different and the ellipsoid is close to a sphere: $b_i = R + \varepsilon B_i$, (i = 1,2,3). Then the equations of motion (2.1) allow of a particular solution of the form (2.4), containing a term with logarithms, if the following condition is not satisfied

$$B_1 = B_2 = B_3$$

and these equations do not allow of an additional integral, functionally independent with the remaining integrals in this solution, that is analytic in ε and algebraic in the phase variables.

The proof of the theorem is similar to that considered previously (Ref. 4, p. 73 – 81).

By introducing the small parameter in this form, we can use Theorem 1 for n = 6 and k = 4.

In the first approximation we obtain the equations

$$\dot{m}_1 = (A_3 - A_2)m_2m_3 + \varepsilon(B_3 - B_2)\gamma_2\gamma_3, \quad \dot{\gamma}_1 = A_3m_3\gamma_2 - A_2m_2\gamma_3 \quad (123)$$
$$A_i = J_i^{-1}, \quad (i = 1, 2, 3) \tag{2.6}$$

These equations are identical with the Kirchhoff–Clebsch equations for the motion of a body in an ideal fluid, for which their integrals have the form

$$H = \frac{1}{2}\left(A_1m_1^2 + A_2m_2^2 + A_3m_3^2\right) + \varepsilon\frac{1}{2}\left(B_1\gamma_1^2 + B_2\gamma_2^2 + B_3\gamma_3^2\right),$$
$$K = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \quad \Gamma = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

We make the replacement

$$\bar{m}_1 = m_1\pi_1; \quad \pi_1 = \sqrt{(A_1 - A_3)(A_2 - A_1)} \quad (123)$$

Note that in this case the following equalities hold

$$\frac{A_1^2}{\pi_1^2} + \frac{A_2^2}{\pi_2^2} + \frac{A_3^2}{\pi_3^2} = -1, \quad \frac{A_1}{\pi_1^2} + \frac{A_2}{\pi_2^2} + \frac{A_3}{\pi_3^2} = 0$$

In the new variables (we henceforth omit the bar over m_i) Eqs (2.6) can be written in the form

$$\dot{m}_1 = m_2m_3 + \varepsilon\pi_1(B_3 - B_2)\gamma_2\gamma_3, \quad \dot{\gamma}_1 = A_3m_3\pi_3^{-1}\gamma_2 - A_2m_2\pi_2^{-1}\gamma_3 \quad (123) \tag{2.7}$$

Equations (2.7) when $\varepsilon = 0$ have four integrals:

$$I_1 = m_1^2 - m_2^2, \quad I_2 = m_3^2 - m_2^2, \quad I_3 = m_1\pi_1^{-1}\gamma_1 +$$
$$+ m_2\pi_2^{-1}\gamma_2 + m_3\pi_3^{-1}\gamma_3, \quad I_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \tag{2.8}$$

where $2H_0 = A_1m_1^2\pi_1^{-2} + A_2m_2^2\pi_2^{-2} + A_3m_3^2\pi_3^{-2}$ is a linear combination of the integrals I_1 and I_2.

Clearly all four integrals are independent.

Unperturbed system (2.7) has the exact the solution

$$m_1^{(0)} = -t^{-1}, \quad m_2^{(0)} = t^{-1}, \quad m_3^{(0)} = t^{-1}, \quad \gamma_1^{(0)} = -\frac{A_1}{\pi_1}, \quad \gamma_2^{(0)} = \frac{A_2}{\pi_2}, \quad \gamma_3^{(0)} = \frac{A_3}{\pi_3}$$

We will obtain a solution of the form (2.4) for Eqs (2.7). To do this, according to the theorem, we obtain the eigenvectors and eigenvalues of the Kovalevskaya matrix. The general solution of Eqs (2.7) has the form

$$\mathbf{x}^{(1)}(t) = \sum_{i=1}^{6} c_i \boldsymbol{\beta}_i t^{\mathbf{p}+\rho_i}$$

Here the coefficients c_i are found by the method of variation of arbitrary constants.

We will write the first three components of the vector $x^{(1)}(t)$

$$m_1^{(1)} = c_1 t + c_2 t - c_3 t^{-2}, \quad m_2^{(1)} = c_1 t + c_3 t^{-2}, \quad m_3^{(1)} = c_2 t + c_3 t^{-2}$$

The Kovalevskaya exponents for them are $R = \{-1,2,2\}$.

The solution in the first approximation for the inhomogeneous case can be found by the method of variation of arbitrary constants and has the form

$$c_1 = \Delta_1 \ln t + c_1^0, \quad c_2 = \Delta_2 \ln t + c_2^0, \quad c_3 = \Delta t^3 + c_3^0,$$

where

$$\Delta_1 = \begin{vmatrix} \zeta_{123} & 1 & -1 \\ \zeta_{231} & 0 & 1 \\ \zeta_{312} & 1 & 0 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 1 & \zeta_{123} & -1 \\ 1 & \zeta_{231} & 1 \\ 0 & \zeta_{312} & 0 \end{vmatrix}, \quad \Delta = \begin{vmatrix} 1 & 1 & \zeta_{123} \\ 1 & 0 & \zeta_{231} \\ 0 & 1 & \zeta_{312} \end{vmatrix}$$

$$\zeta_{ijk} = (B_k - B_j)\gamma_k \gamma_j \pi_i, \quad i, j, k = 1, 2, 3 \tag{2.9}$$

and c_1^0, c_2^0, c_3^0 are arbitrary constant.

In this case the first three components of the vector s_{11}, which occurs in expansion (2.4), specifying the exact solution, have the form

$$(\mathbf{s}_{11}(t))_1 = (\Delta_1 + \Delta_2)t, \quad (\mathbf{s}_{11}(t))_2 = \Delta_1 t, \quad (\mathbf{s}_{11}(t))_3 = \Delta_2 t$$

By virtue of the necessary condition of integrability (2.5), it is necessary to take the scalar product of the vectors s_{11} and $(1, -1, 0, 0, 0, 0)$, $(0, 1, -1, 0, 0, 0)$.

Hence, condition (2.5) is equivalent to equating to zero the coefficients of the logarithms

$$-\zeta_{123} = \zeta_{231} = \zeta_{312} \tag{2.10}$$

System (2.10) can be rewritten in the form

$$\frac{(B_2 - B_3)A_2}{A_3 - A_2} + \frac{(B_1 - B_3)A_1}{A_1 - A_3} = 0 \quad (123)$$

and then in the form of the single Clebsch condition

$$\frac{B_2 - B_3}{A_1} + \frac{B_3 - B_1}{A_2} + \frac{B_1 - B_2}{A_3} = 0$$

We will now consider the second approximation of Eqs (2.1)

$$\begin{aligned} \dot{m}_1 &= m_2 m_3 + \varepsilon\zeta_{123} + \varepsilon^2 \left(\langle \mathbf{P}(\gamma)\mathbf{m}, \mathbf{m} \rangle + \frac{mg\zeta_{123}}{R\pi_1^2} \right) \\ \dot{\gamma}_1 &= A_3 \pi_3^{-1} m_3 \gamma_2 - A_2 m_2 \pi_2^{-1} \gamma_3 + \varepsilon^2 \langle \mathbf{q}(\gamma), \mathbf{m} \rangle \quad (123) \end{aligned} \tag{2.11}$$

Here we have introduced the notation: P – a certain matrix, which depends on γ, and $\mathbf{q}$ a certain vector which depends on γ,

$$\kappa_{123} = (B_2 - B_3)\gamma_2\gamma_3 \left[(B_2 - B_1)(2\gamma_2^2 - 1) + (B_3 - B_1)(2\gamma_3^2 - 1) \right] \pi_1 \quad (123)$$

The theorem is again applicable, since the logarithm can now occur in the second approximation. The coefficients of the terms, quadratic in $\mathbf{m}$, of Eqs (2.11) are not important, since they will be proportional to t^{-2} and hence the solutions in (2.9) will have no logarithmic part. Carrying out a discussion similar to the case of the first approximation, we obtain the conditions for the integral to exist

$$-\kappa_{123} = \kappa_{231} = \kappa_{312} \tag{2.12}$$

From Eqs (2.10) and (2.12) we have the equations

$$\kappa_{123}\zeta_{123}^{-1} = \kappa_{231}\zeta_{231}^{-1} = \kappa_{312}\zeta_{312}^{-1}$$

They give two relations connecting the parameters B_1, B_2 and B_3. Simplifying them, taking into account the equality $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = -1$, it can be shown that they are equivalent to the condition $B_1 = B_2 = B_3$.

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